# AI Enabled Control Engineering

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# Recap

- What is control?
- Why study control?
- The inverted pendulum as a classical example
- Block diagram
- Open loop, closed-loop, and negative feedback
- Input, output, and state

# Lecture 2

- What is a linear dynamical system?
- Why do/can we focus on linear systems?
- From nonlinear cart-pole to a linear model
- Laplace transform and its properties.
- Transfer function.

# Why do we focus on linear systems?

Real engineering systems are often nonlinear.

- The cart-pole dynamics are nonlinear and coupled.
- Many systems can be approximated by linear models near an operating point.
- Linear systems admit clean analysis in time domain and frequency domain.
- Linear models are the starting point of many classical control methods.

From nonlinear to linear is one of the most important simplifications in control.

# Linear dynamical system

Consider system of linear equations

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_n,$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_n,$$

$$\vdots$$

$$\dot{x}_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_n,$$

can be written in matrix form as $\dot{x} = Ax$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

# From nonlinear cart-pole to a linear model

For the cart-pole system, define the state as

$$x = \begin{bmatrix} p & \dot{p} & \theta & \dot{\theta} \end{bmatrix}^\top .$$

The full system is nonlinear:

$$\dot{x} = f(x, u).$$

Around the upright equilibrium, we can linearize it as

$$\dot{x} = Ax + Bu.$$

This is why linear system theory is relevant even when the original system is nonlinear.

# Cart-pole nonlinear dynamics model

For the Gymnasium CartPole model:

$$M = 1.0 \text{ kg}, \qquad m = 0.1 \text{ kg}, \qquad l = 0.5 \text{ m},$$

where $M$ is the cart mass, $m$ is the pole mass, and $l$ is the distance from the pivot to the pole center of mass.

The coupled nonlinear equations can be written as

$$(M + m)\ddot{p} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = u,$$

$$\left(I_{\mathrm{cm}} + ml^2\right)\ddot{\theta} = mgl\sin\theta - ml\cos\theta\,\ddot{p}.$$

For a uniform rod,

$$I_{\mathrm{cm}} = \frac{1}{3}ml^2,$$

hence

$$I_{\mathrm{cm}} + ml^2 = \frac{4}{3}ml^2.$$

# Cart-pole update equations in simulation

In the simulation code, the nonlinear accelerations are computed as

$$\mathrm{temp} = \frac{u + ml\dot{\theta}^2 \sin\theta}{M+m},$$

$$\ddot{\theta} = \frac{g\sin\theta - \cos\theta\,\mathrm{temp}}{l\left(\frac{4}{3} - \frac{m\cos^2\theta}{M+m}\right)},$$

$$\ddot{p} = \mathrm{temp} - \frac{ml\cos\theta}{M+m}\ddot{\theta}.$$

Then the continuous-time dynamics are discretized by Euler update with sampling time

$$\tau = 0.02\ \mathrm{s}.$$

This gives a concrete example of how a physical model enters a control environment.

# Linearized Cart-Pole Dynamics Around Upright Equilibrium

Around the upright equilibrium,

$$\theta \approx 0, \qquad \dot{\theta} \approx 0,$$

the nonlinear cart-pole system can be linearized.

For general parameters $M, m, l, g$, the linearized equations are

$$\ddot{p} = -\frac{3mg}{4M+m}\theta + \frac{4}{4M+m}u,$$

$$\ddot{\theta} = \frac{3(M+m)g}{l(4M+m)}\theta - \frac{3}{l(4M+m)}u.$$

# State-Space Form of the Linearized Cart-Pole Model

Define the state vector as

$$x = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}.$$

Then the linearized system can be written in state-space form:

$$\dot{x} = Ax + Bu.$$

Specifically,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{3mg}{4M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{3(M+m)g}{l(4M+m)} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \dfrac{4}{4M+m} \\ 0 \\ -\dfrac{3}{l(4M+m)} \end{bmatrix}.$$

# Why Laplace transform?

How do we simplify ODEs?

The Laplace transform converts differential equations into algebraic equations:

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt.$$

Why is this useful?

- differentiation becomes multiplication by $s$
- system response can be described in the $s$-domain
- poles and zeros become visible
- it naturally supports causal systems

# Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad \text{Laplace Integral, Q: why?}$$

1. Conservation;
2. Causality:
   The relationship between an event (the cause) and a second event (the effect), where the second event is understood as a consequence of the first;
   Current state ($x(t)$) and output ($y(t)$) depending on past input ($u(\tau)$ for $\tau \leq t$) is causal;
   Current state (and output) depending on future input is anti-causal.

# Laplace transform

1. $\mathcal{L}[tf(t)] = -F'(s)$
2. $\mathcal{L}[e^{-at}f(t)] = F(s+a)$ — Frequency Shift
3. $\mathcal{L}[f(t-T)] = e^{-sT}F(s)$ — Time Shift
4. $\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$ — Scaling
5. $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^-)$ — Differentiation
6. $\mathcal{L}\left[\int_{0^-}^{t} f(\tau)\, d\tau\right] = \frac{F(s)}{s}$ — Integration Theorem
7. $f(\infty) = \lim_{s\to 0}(sF(s))$ — Final Value Theorem
8. $f(0^+) = \lim_{s\to\infty}(sF(s))$ — Initial Value Theorem

# Laplace table

| Representative input | Laplace Transform |
| --- | --- |
| $\delta(t)$ | $1$ |
| $u(t)$ | $\frac{1}{s}$ |
| $tu(t)$ | $\frac{1}{s^2}$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ |

**Table:** Table of Laplace Transforms

# Laplace table

| | |
|---|---|
| $\sin(\omega t)u(t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos(\omega t)u(t)$ | $\frac{s}{s^2+\omega^2}$ |
| $e^{-kt}\sin(\omega t)u(t)$ | $\frac{\omega}{(s+k)^2+\omega^2}$ |
| $e^{-kt}\cos(\omega t)u(t)$ | $\frac{s+k}{(s+k)^2+\omega^2}$ |

**Table:** Table of Laplace Transforms

Q: What is the previous state space model using Laplace transform?

# Derivative property

For a signal $z(t)$,

$$\mathcal{L}[\dot{z}(t)] = sZ(s) - z(0),$$

and

$$\mathcal{L}[\ddot{z}(t)] = s^2 Z(s) - sz(0) - \dot{z}(0).$$

For the cart-pole model, this is especially useful because the dynamics are written in terms of

$$\ddot{p} \qquad \text{and} \qquad \ddot{\theta}.$$

Under zero initial condition,

$$\mathcal{L}[\ddot{z}(t)] = s^2 Z(s).$$

Therefore, the second-order differential equations in time domain can be converted into algebraic equations in the $s$-domain.

# Transfer function

After taking Laplace transform, differential equations become algebraic equations in the $s$-domain.
For a linear time-invariant system, we would like to describe the relation between input $U(s)$ and output $Y(s)$ directly.

Under zero initial condition, the transfer function is defined as

$$G(s) = \frac{Y(s)}{U(s)}.$$

It provides a compact input-output description of the system, and it is especially useful for analyzing poles, zeros, and system response.

# From time domain to frequency domain

For the linearized cart-pole model, the time-domain equations are

$$\ddot{p} = -\frac{3mg}{4M+m}\theta + \frac{4}{4M+m}u,$$

$$\ddot{\theta} = \frac{3(M+m)g}{l(4M+m)}\theta - \frac{3}{l(4M+m)}u.$$

Taking Laplace transform under zero initial condition, we obtain

$$s^2 P(s) = -\frac{3mg}{4M+m}\Theta(s) + \frac{4}{4M+m}U(s),$$

$$s^2\Theta(s) = \frac{3(M+m)g}{l(4M+m)}\Theta(s) - \frac{3}{l(4M+m)}U(s).$$

The equations in the $s$-domain are the starting point for deriving transfer functions.

# Transfer functions of the linearized Cart-Pole

From

$$s^2\Theta(s) = \frac{3(M+m)g}{l(4M+m)}\Theta(s) - \frac{3}{l(4M+m)}U(s),$$

we obtain

$$\frac{\Theta(s)}{U(s)} = -\frac{3}{l(4M+m)} \cdot \frac{1}{s^2 - \dfrac{3(M+m)g}{l(4M+m)}}.$$

Substituting this into the equation for $P(s)$ gives

$$\frac{P(s)}{U(s)} = \frac{\dfrac{4}{4M+m}s^2 - \dfrac{3g}{l(4M+m)}}{s^2\left(s^2 - \dfrac{3(M+m)g}{l(4M+m)}\right)}.$$

# Numerical transfer functions of Cart-Pole

For the Gymnasium CartPole model,

$$M = 1.0\,\mathrm{kg}, \qquad m = 0.1\,\mathrm{kg}, \qquad l = 0.5\,\mathrm{m}, \qquad g = 9.8\,\mathrm{m/s^2}.$$

Substituting these values into the linearized model gives

$$\frac{\Theta(s)}{U(s)} = -\frac{1.4634}{s^2 - 15.7756},$$

$$\frac{P(s)}{U(s)} = \frac{0.9756\,s^2 - 14.3415}{s^2(s^2 - 15.7756)}.$$

- input $\rightarrow$ pole angle: $\frac{\Theta(s)}{U(s)}$
- input $\rightarrow$ cart position: $\frac{P(s)}{U(s)}$

# What can we do with transfer functions?

Once the transfer functions are obtained, we can

- identify poles and zeros,
- understand the input-output behavior,
- compare the cart-position channel and the pole-angle channel,
- predict step and impulse responses,
- prepare for stability analysis and closed-loop design.

In the next lecture, we will use these ideas to study stability and closed-loop behavior.

# Summary

- Linear system.
- From nonlinear to linear.
- Cart-pole nonlinear dynamics and linearization around the upright equilibrium.
- Laplace transform and its useful properties.
- Transfer function as an input-output description in the $s$-domain.
- Transfer functions of the linearized cart-pole.
- Preparation for stability analysis and closed-loop design.