AI Enabled Control Engineering

Professor Xun Huang

Aeronautics and Astronautics
College of Engineering
Peking University

huangxun@pku.edu.cn
https://xunger99.github.io/xunger/

Recap

- From nonlinear cart-pole to a linear model.
- Laplace transform and its useful properties.
- Transfer function as an input-output description.
- Two channels of the linearized cart-pole:

$$\frac{\Theta(s)}{U(s)}, \qquad \frac{P(s)}{U(s)}.$$

Lecture 3

- Closed-loop transfer function.
- Characteristic equation.
- Poles and zeros.
- Stability and marginal stability.
- Routh criterion.
- Root locus.

Why is the open-loop cart-pole unstable?

Consider the pole-angle channel

$$\frac{\Theta(s)}{U(s)} = -\frac{1.4634}{s^2 - 15.7756}.$$

Assume zero initial condition and apply a unit-step input

$$u(t) = 1, \qquad U(s) = \frac{1}{s}.$$

Then

$$\Theta(s) = -\frac{1.4634}{s(s^2 - 15.7756)}.$$

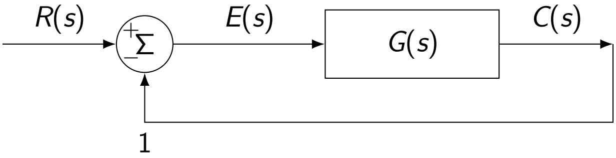
Hence

$$\theta(t) \approx -0.0928\big(\cosh(3.97t) - 1\big).$$

The response diverges. Therefore, the upright cart-pole is open-loop unstable.

Closed-loop transfer function

Consider a standard negative-feedback system with unit feedback:



We have

$$E(s) = R(s) - C(s), \qquad C(s) = G(s)E(s).$$

Therefore,

$$C(s) = G(s)\big(R(s) - C(s)\big).$$

Hence, the closed-loop transfer function is

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}.$$

Closed-loop cart-pole angle channel

Let us use the pole-angle channel as the plant:

$$G_\theta(s) = \frac{\Theta(s)}{U(s)} = -\frac{1.4634}{s^2 - 15.7756}.$$

Suppose a proportional controller is used:

$$u(t) = -K\theta(t).$$

Then the closed-loop transfer function becomes

$$T_\theta(s) = \frac{KG_\theta(s)}{1 + KG_\theta(s)}.$$

Substituting $G_\theta(s)$,

$$T_\theta(s) = \frac{-1.4634K}{s^2 - 15.7756 - 1.4634K}.$$

Characteristic equation

For a transfer function

$$T(s) = \frac{N(s)}{D(s)},$$

the equation

$$D(s) = 0$$

is called the characteristic equation.
For a unity-feedback system

$$T(s) = \frac{G(s)}{1 + G(s)},$$

the characteristic equation is

$$1 + G(s) = 0.$$

This is one of the most important equations in classical control, because stability is determined by its roots.

Poles and zeros

For

$$G(s) = \frac{N(s)}{D(s)},$$

we define

- Zero: root of $N(s) = 0$
- Pole: root of $D(s) = 0$
- Characteristic equation: $D(s) = 0$

Physical interpretation:

- poles determine the main system dynamics,
- zeros modify the input-output response shape,
- closed-loop design mainly aims at changing the pole locations.

Definitions of stability

Stability of the equilibrium of a dynamic system

- Lyapunov stable, if for every $\epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$, such that

$$\|x(0) - x_e\| < \delta, \ \ \|x(t) - x_e\| < \epsilon, \ \ t \geq 0$$

- Asymptotic stable, $\exists \delta > 0$, such that $\|x(0) - x_e\| < \delta$, then

$$\lim_{t \to \infty} \|x(t) - x_e\| = 0.$$

- Exponentially stable, $\exists \alpha, \beta, \delta > 0$, such that $\|x(0) - x_e\| < \delta$, then

$$\|x(t) - x_e\| \leq \alpha \|x(0) - x_e\| e^{-\beta t}.$$

BIBO stability and its relation to other notions

BIBO stability :

- if every bounded input produces a bounded output,
- then the system is BIBO stable.

That is,

$$\|u(t)\| < M_u < \infty \text{ for all } t \quad \Longrightarrow \quad \|y(t)\| < M_y < \infty \text{ for all } t.$$

Relation to other notions of stability:

- Lyapunov, asymptotic, and exponential stability describe the behavior of the **state**.
- BIBO stability describes the input-output behavior of the **system**.
- For linear systems, asymptotic stability usually implies BIBO stability.
- But BIBO stability and internal state stability are not exactly the same concept.

Root locations and stability

A necessary condition for a system to be stable is that the real parts of the roots of the characteristic equation are negative.

If the real part is negative:

$$e^{st} \to 0 \quad \text{as } t \to \infty.$$

If the real part is zero, the system may be marginal.
If the real part is positive:

$$e^{st} \to \infty,$$

and the response grows.

So pole locations in the complex plane give a direct stability picture.

Cart-pole as a stability example

For the angle channel

$$\frac{\Theta(s)}{U(s)} = -\frac{1.4634}{s^2 - 15.7756},$$

the characteristic equation is

$$s^2 - 15.7756 = 0.$$

The roots are

$$s = \pm 3.97.$$

Thus:

- one pole lies in the left-half plane,
- one pole lies in the right-half plane.

Therefore, the open-loop upright cart-pole is unstable.

Routh stability criterion

Consider the following equation, which is a generalisation of a characteristic equation;

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

The criterion is applied through using what is called the **Routh Table.**

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_1 & b_2 & b_3 & \ldots \\
s^{n-3} & c_1 & c_2 & c_3 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s & \ldots & \ldots & \ldots & \ldots \\
1 & \ldots & \ldots & \ldots & \ldots
\end{array}$$

Routh stability criterion

Where in this array

$$a_n, a_{n-1}, a_{n-2}, \ldots$$

are the coefficients of the equation, and

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}},$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}},$$

$$c_1 = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1},$$

$$c_2 = \frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1}.$$

Routh stability criterion

- The table is continued horizontally and vertically until only zeros are obtained.
- All roots of the characteristic equation have negative real parts if and only if the elements of the **first column** of the Routh Table have the **same sign**.
- Otherwise the number of roots with positive real parts is equal to the number of changes of sign.

A simple Routh example

Consider

$$s^3 + 6s^2 + 12s + 8 = 0.$$

The Routh table is

$$\begin{array}{c|ccc} s^3 & 1 & 12 & 0 \\ s^2 & 6 & 8 & 0 \\ s^1 & \frac{64}{6} & 0 & 0 \\ s^0 & 8 & 0 & 0 \end{array}$$

All entries in the first column are positive.

Therefore, all roots have negative real parts, so the system is stable.

Routh stability criterion

Often you need to find a range of values of a particular parameter for which the system has stable roots. In the following example this will be demonstrated.
Example:

$$s^3 + 3s^2 + 3s + 1 + K = 0.$$

$$\begin{array}{c|ccc} s^3 & 1 & 3 & 0 \\ s^2 & 3 & 1+K & 0 \\ s^1 & \dfrac{8-K}{3} & 0 & 0 \\ s^0 & 1+K & 0 & 0 \end{array}$$

Q: What is k for stability?

Root locus concept

The above method tells stability for a fixed system. But the closed-loop gain K can vary.
How do we analyze the stability of the whole closed-loop system as K changes?

Root locus is a graphical method for examining how the roots of the characteristic equation change with variation of the feedback gain.

In other words:

- as K varies from 0 to ∞,
- the closed-loop poles move along certain trajectories in the complex plane.

Root locus method

Preliminary knowledge

Given

$$F(s) = \frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{N}(s+p_j)},$$

which can be written to a magnitude and argand form;

$$F(s) = Me^{i\theta}, \text{ then}$$

$$M = \frac{\prod \text{ zero lengths}}{\prod \text{ pole lengths}}, \qquad \theta = \sum_{i=1}^{M} \angle(s+z_i) - \sum_{j=1}^{N} \angle(s+p_j).$$

Root locus method

Starting points

Here

$$F(s) = 1 + KG(s), \text{ where } G(s) = \frac{N_G(s)}{D_G(s)}.$$

That is

$$D_G(s) + KN_G(s) = 0, \text{ as } K \to 0, \therefore D_G(s) = 0.$$

Hence, the root locus starts at the poles of $G(s)$, i.e. the poles of open loop system.

Ending points

$$D_G(s) + KN_G(s) \approx KN_G(s), \text{ as } K \to \infty,$$

the ending points are at

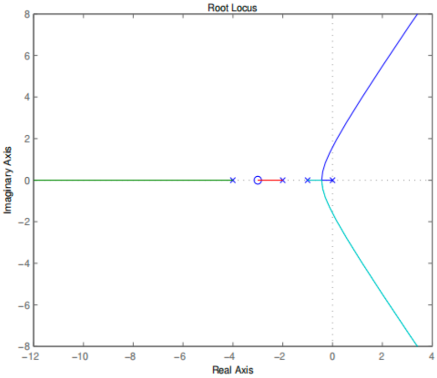
$$N_G(s) = 0,$$

i.e. the zeros of open loop system.

Root locus method

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}, \qquad H(s) = 1$$

The MATLAB code can help us to do root locus,



Closed-loop poles of the pole-angle channel

For the pole-angle channel with proportional feedback,

$$u(t) = -K\theta(t),$$

the closed-loop transfer function is

$$T_\theta(s) = \frac{-1.4634K}{s^2 - 15.7756 - 1.4634K}.$$

Therefore, the closed-loop characteristic equation is

$$s^2 - 15.7756 - 1.4634K = 0,$$

so the closed-loop poles are

$$s = \pm\sqrt{15.7756 + 1.4634K}.$$

As K varies, these poles move along the real axis, and pure proportional feedback does not move both poles into the left-half plane.

Summary

- Closed-loop transfer function.
- Characteristic equation.
- Poles and zeros.
- Stability, marginal stability, and BIBO stability.
- Routh stability criterion.
- Root locus as a tool for gain-dependent pole movement.